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Newton's Second Law on Positive Real Line $\mathbb{R}_+$

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ABSTRACT

In this work, a classical particle that moves on $\mathbb{R}_+$ will be investigated by using the affine commutator relation which corresponds to the Poisson bracket in classical mechanics where the equation of motions and the Newton's second law will be modified. Two example of physical systems will be studied further using the modified Newton's second law namely free particle and harmonic oscillator. The result concludes that for a free particle, the force on $\mathbb{R}_+$ is no longer zero and a linear force due positive real line and fixed momentum. While, classical dynamics on $\mathbb{R}_+$ for simple harmonic oscillator system is a qubic force.

Keywords: classical mechanics, positive real line, Newton's second law

1. INTRODUCTION

The positive real line is a configuration space $Q = \mathbb{R}_+$ that restricts a particle to move on the $(0, \infty)$ region of the real line $\mathbb{R}$. Quantization on $\mathbb{R}$ can be done via canonical quantization, where the position $\hat{q}$ and momentum $\hat{p}$ defined as self-adjoint operators acting on a Hilbert space, and satisfy the following canonical commutation relation

$$[\hat{q}, \hat{p}] = i\hbar, \tag{1}$$

where ħ is the Planck constant. However, when we proceed to quantize a particle moving on $\mathbb{R}_+$, apparently the momentum operator $\hat{p}$ is no longer self-adjoint operator (Al-Hashimi & Wiese, 2021a), and does not qualify as a physical observable. Consequently, canonical quantization fails due to the momentum operator is not well-defined on $\mathbb{R}_+$. Alternatively, it will be replaced by a more appropriate quantum momentum operator which is denoted by $\hat{\pi}$ on $\mathbb{R}_+$ as it was introduced by affine quantization (Klauder, 1999a).

Quantum operators in canonical quantization are $-\infty < \hat{q} < \infty$,, and $-\infty < \hat{p} < \infty$, while quantum operator for affine quantization is given by $\hat{x} > 0$ and $-\infty < \hat{\pi} < \infty$, which then make the affine commutation relation;

$$[\hat{x}, \hat{\pi}] = i\hbar\hat{x}. \tag{2}$$

Several works on the affine quantization of a quantum theory on $\mathbb{R}_+$ has already been widely explored. For example in a quantum system that involves gravity (Isham & Kakas, 1984;

Isham, 1984; Klauder, 1999b) a free particle (Gouba, 2020), a particle in a box (Al-Hashimi & Wiese, 2021b), covariant scalar field $(\phi^4)_4$ (Riccardo & Klauder, 2021), and oscillator (Gouba, 2020; Isiaka & Gouba, 2021). Note that, the quantization starts with the classical observables in the Poisson bracket to the commutator of the quantum operators. However, there is a lack of study in any classical theory of the positive real line. Some authors have developed the classical system by defining the symplectic structure which corresponds to the commutation relations. For instance, Romero et al. (2003) as well as Acatrinei (2005) studied classical dynamics based on the noncommutative space, which was further incorporated with a magnetic field background (Acatrinei, 2005; Djemai, 2004). Meanwhile Gao-Feng et al. (2008) delved into noncommutative phase space, and Chung (2006) employed the minimal length uncertainty principle. These studies collectively revealed that linear extended variables in noncommutative systems and minimal length uncertainty principle provide some corrections to the Newton's second law. There exists a notable research gap in the current literature concerning the implications of affine commutation relations on classical dynamics. Therefore this study aims to bridge this gap by providing a detailed analysis of the classical dynamics based on the affine commutation relations, offering a comprehensive solution to the research gap and contributing valuable insights, particularly in the context of classical mechanics.

The purpose of this work is to study a classical particle that moves on $\mathbb{R}_+$ using the generalized Poisson bracket corresponding to the affine commutation relation (2). The equation of motion is used to derive the Newton's second law, and later will be applied to some examples namely a free particle and simple harmonic oscillators. The organization of this paper is such in Sec. 2, we derive the equation of motion based on the Poisson brackets corresponding to affine commutation relation (2). In Sec. 3 we applied our results in Sec. 2 for some examples, and lastly the conclusion is given in Sec. 4.

2. AFFINE CLASSICAL MECHANICS

In the classical limit $\hbar \to 0$, the commutator for quantum operator is replaced by the Poisson bracket for the corresponding classical variables. Let A, B develops

$$\frac{1}{i\hbar}\left[\hat{A},\hat{B}\right] \to \{A,B\}.$$

Then the Poisson bracket is consistent with the affine commutation relation (2) is given by

$$\{x,\pi\} = x. \tag{3}$$

The affine Poisson bracket (3) can also be obtained from the canonical Poisson bracket$\{q,p\} = 1$, by simply manipulating the coordinates $x \to q$ and $\pi \to qp$ as in (Klauder, 2012) as follows;

$$\{x,\pi\} \equiv \{q,qp\} = q\{q,p\} = q \cdot 1 \equiv x.$$

Geometrically, global coordinates between real line $\mathbb{R}$ and positive real line $\mathbb{R}_+$ can be obtained via diffeomorphism map which denoted by i^* and vice versa j^*, where the canonical variable pair (q,p) and affine variable pair (x,π) are linked in the following morphisms

$$i^*(q,p) = (e^q, e^{-q}p) \equiv (x,\pi), \tag{4a}$$
$$j^*(x,\pi) = (\log x\,, x^{-1}\pi) \equiv (q,p). \tag{4b}$$

Here, the variable π is also known as dilation variable (Klauder, 1999, Gouba, 2020 and Isiaka & Gouba, 2021).

Hamiltonian $H(q,p)$ function is now replaced by the affine variables (4) by introducing $p \to \sqrt{\alpha}\pi$ in the Hamiltonian $H(x,\pi)$ such that

$$H(x,\pi) = \alpha\frac{\pi^2}{2m} + V(x), \tag{5}$$

where the dimension of α is $[\text{Length}]^{-2}$. Moreover, the kinetic part in (5) is modified to be as $K = \alpha\frac{\pi^2}{2m}$ instead of $K = \frac{1}{x^2}\frac{\pi^2}{2m}$, while the potential part $V(x)$ still unchanged.

In general, the Hamilton's equation (equation of motion) of the classical system are defined with Hamiltonian H as follows;

$$\dot{A} \coloneqq \{A, H\}.$$

Thus the equation of motions becomes;

$$\dot{x} = \{x, H\} = \frac{\alpha\pi}{m}x, \tag{6a}$$
$$\dot{\pi} = \{\pi, H\} = -x\frac{d}{dx}V(x) \tag{6b}$$

The dynamical system is obtained from the doubly Poisson brackets (6a) where the acceleration $\ddot{x}$ is given by

$$\ddot{x} = \big\{\{x, H\}, H\big\}, \tag{7}$$

and the Newton's second law $F = m\ddot{x}$ gives

$$F = \frac{\alpha^2\pi^2}{m}x - \alpha x^2\frac{d}{dx}V(x). \tag{8}$$

The force in terms of the rate of the coordinates (6) becomes

$$F = \alpha(\dot{x}\pi + x\dot{\pi}). \tag{9}$$

The result shows that the force on $\mathbb{R}_+$ is the rate of xπ as follows

$$F = \alpha\frac{d}{dt}(x\pi). \tag{10}$$

This expression provides valuable insights into the relationship between the force and the rate of change of the coordinates in this classical system.

3. RESULTS AND DISCUSSION

This work proceeds further with two example of the classical systems such as a free particle and harmonic oscillator.

3.1 Free Particle

Consider the free Hamiltonian where $V(x) = 0$ is given as

$$H(x,\pi) = \alpha\frac{\pi^2}{2m}. \tag{11}$$

Thus the equation of motion (6) becomes

$$F = \frac{\alpha^2 \pi^2}{m} x. \tag{12}$$

Consequently, the result shows that the force of a free particle along $\mathbb{R}_+$ is no longer zero. In addition, the result in (12) apparently is a linear force due to positive real line x if π is a fixed variable.

3.2 Harmonic Oscillator

The harmonic oscillator system is a simple example and has been subjected to many works. Now, consider the simple harmonic oscillator system for which Hamiltonian is given by

$$H(x, \pi) = \alpha \frac{\pi^2}{2m} + \frac{m\omega^2}{2} x^2, \tag{13}$$

with mass m and angular frequency ω of the harmonic oscillator. The equation of motion for the case that we are considering is given by;

$$F = \frac{\alpha^2 \pi^2}{m} x - \alpha m \omega^2 x^3. \tag{14}$$

The result shows that the classical dynamics of the harmonic oscillator system on $\mathbb{R}_+$ is a qubic equation with fixed momentum π.

For the free particle, the obtained force (12) indicates that when positioned along $\mathbb{R}_+$, it experiences a non-zero linear force proportional to x when π is held constant. In the case of the harmonic oscillator, the force (14) shows a cubic equation with a fixed momentum π. These results provide valuable insights into the behavior of these systems under the given conditions.

4. CONCLUSION

We have studied the laws of motion of classical particles along the positive real line $\mathbb{R}_+$. Classical dynamics on $\mathbb{R}_+$ for a free particle and simple harmonic oscillator respectively are linear force and qubic force due to positive real line x, which fixed momentum π. Compare to the existing literature, noncommutative systems extend the force variable by some correction terms. Furthermore, the study of classical dynamics on $\mathbb{R}_+$ is noteworthy that a force emerges in the case of the free particle, providing further insights into the intricate dynamics of these classical systems on the positive real line. In practice, the application of $\mathbb{R}_+$ can be geometrically employed to study gravity in systems where the parameter $r > 0$.

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