On The Exterior Degree of Some Finite *p*-Groups

Darjah Peluaran bagi Beberapa Kumpulan-p Terhingga

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Abstract

In any finite group G, the commutativity degree of G (denoted by $P(G)$) is the probability that two randomly chosen elements of G commute. $P(G)$ has been introduced by Erdos and Turan (1968) for symmetric groups. Gustafson (1973) did the same for compact groups while MacHale (1974) determined the commutativity degree of finite rings. The exterior square of G, denoted as $G \wedge G$, is defined as $G \wedge G = G \otimes G / \nabla(G)$ where $(G \otimes G)$ is the nonabelian tensor square of G and $\nabla(G)$ is a subgroup of G generated by $x \otimes x$ for all $x \in G$. Recently, Niroomand and Rezaei (2011) give some relations between the concept of exterior square and commutativity degree by defining the exterior degree of a finite group G, denoted as $P^\wedge(G)$, to be the probability for two elements g and g' in G such that $g \wedge g' = 1$ where $1_\wedge$ is the identity of $G \wedge G$. In this paper, the exterior degree of some finite groups is presented.

Keywords   commutativity degree, exterior square, finite group

Abstrak

Dalam mana-mana kumpulan terhingga G, darjah kekalisan tukar tertib bagi G (diwakili oleh $P(G)$) adalah kebarangkalian bagi dua unsur yang dipilih secara rawak dalam G adalah kalis tukar tertib. $P(G)$ telah diperkenalkan oleh Erdos dan Turan (1968) untuk kumpulan simetri. Gustafson (1973) melakukan perkara yang sama untuk kumpulan padat manakala MacHale (1974) menentukan darjah kekalisan tukar tertib bagi gelanggang yang terhingga. Kuasa dua peluaran bagi G diwakili sebagai $G \wedge G$, boleh ditakrifkan sebagai $G \wedge G = G \otimes G / \nabla(G)$ yang mana $G \otimes G$ adalah kuasa dua tensor tak abelan bagi G dan $\nabla(G)$ adalah subkumpulan G yang dijana oleh $x \otimes x$ untuk semua $x \in G$. Baru-baru ini, Niroomand dan Rezaei (2011) memberi beberapa hubungan antara konsep kuasa dua peluaran dan darjah kekalisan tukar tertib dengan mendefinisikan darjah peluaran bagi satu kumpulan terhingga G, diwakili sebagai $P^\wedge(G)$, sebagai kebarangkalian bagi dua unsur g dan g' dalam G yang mana $g \wedge g' = 1$, dan $1_\wedge$ adalah identiti bagi $G \wedge G$. Dalam kertas kerja ini, darjah luaran bagi beberapa kumpulan terhingga dibentangkan.

Katakunci   darjah kekalisan tukar tertib, kuasa dua peluaran, kumpulan terhingga

Introduction

The commutativity degree of a finite group G, denoted as $P(G)$, is the probability that two elements of the group G commute. This can also be defined as the ratio

$$P(G) = \frac{\left|\left\{(x,y) \in G \times G \mid xy = yx\right\}\right|}{|G|^2}. \tag{1}$$

It is obvious that G is abelian if and only if $P(G) = 1$ (Erdos & Turan, 1968). Furthermore, for every nonabelian group G, the commutativity degree of G cannot be arbitrarily close to 1. It will always be less than or equal to 5/8 (MacHale, 1974).

The nonabelian tensor square is a special case of the nonabelian tensor product. The nonabelian tensor square of a group G is generated by symbols $g \otimes h$ where $g, h \in G$ subject to the relations $gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$ for all $g, g', h, h' \in G$ where ${}^g g' = gg'g^{-1}$. Considering the epimorphism mapping κ, where $\kappa : G \otimes G \to G$ defined by $\kappa(g \otimes h) = [g,h]$, and $[g,h] = ghg^{-1}h^{-1}$, the kernel of this mapping is denoted as $J(G)$ and $J(G)$ is a G-trivial subgroup of $G \otimes G$ contained in its center. The subgroup of $J(G)$ generated by the elements $x \otimes x$ for $x \in G$ is denoted by $\nabla(G)$. The exterior square of G is defined as $G \wedge G = (G \otimes G)/\nabla(G)$ while the Schur multiplier of G is $M(G) = J(G)/\nabla(G)$ (Bacon & Kappe 2003).

Recently, Niroomand and Rezaei (2011) give some relations between the concept of exterior square and commutativity degree by introducing the notion of exterior degree of a finite group G, denoted as $P^{\wedge}(G)$, which is the probability for two elements x and y in G such that $x \wedge y = 1_{\wedge}$, where $1_{\wedge}$ is the identity of $G \wedge G$. In mathematical symbols, this notion can be written as

$$P^{\wedge}(G) = \frac{\left|\left\{(x,y) \in G \times G \mid x \wedge y = 1_{\wedge}\right\}\right|}{|G|^2}. \tag{2}$$

Some Preparatory Results

The basic definitions and some concepts of exterior degree for a finite group that will be used throughout this paper are included in this section.

Definition 2.1. (Brown *et al.*, 1987)

If G is a group, then the nonabelian tensor square, $G \otimes G$ is the group generated by the symbols $g \otimes h$ and defined by the relations.

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \tag{3}$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h') \tag{4}$$

for all $g, g', h, h' \in G$ where ${}^h g = hgh^{-1}$ denotes the conjugate of g by h.

Definition 2.2. (Brown *et al*., 1987)

Let $\kappa : G \otimes G \to G'$ defined by $\kappa(g \otimes h) = [g, h]$, and $[g, h] = ghg^{-1}h^{-1}$, where $g \otimes h \in G \otimes G$. Then $J(G) = \ker(\kappa)$.

Definition 2.3. (Brown *et al*., 1987)

Let $\nabla(G)$ denote the subgroup of $J(G)$ generated by the elements $x \otimes x$ for $x \in G$. Thus, $\nabla(G) = \langle (x \otimes x) | x \in G \rangle \subseteq J(G)$.

Definition 2.4. (Brown *et al*., 1987)

The exterior square of G is denoted a $G \wedge G = (G \otimes G) / \nabla(G)$.

Definition 2.5. (Niroomand & Rezaei, 2011)

Let G be a finite group. Then, the exterior degree of G is defined by the ratio

$$P^{\wedge}(G) = \frac{\left| \{ (x, y) \in G \times G \mid x \wedge y = 1_{\wedge} \} \right|}{|G|^2}. \tag{5}$$

Let C be a set defined by $C = \{ (x, y) \in G \times G \mid x \wedge y = 1_{\wedge} \}$. Then we can rewrite the

exterior degree of G by $P^{\wedge}(G) = \dfrac{|C|}{|G|^2}$. The number of elements of C is the size of the

exterior centralizer of x in G. Therefore $|C| = \sum_{x \in G} \left| C_G^{\wedge}(x) \right|$.

Definition 2.6. (Mohd Ali & Sarmin, 2007)

For any group G, the exterior centralizer is defined as $C_G^{\wedge}(x) = \{ a \in G \mid a \wedge x = 1_{\wedge} \}$ or equivalently $C_G^{\wedge}(x) = \{ a \in G \mid a \otimes x = \nabla(G) \}$.

Lemma 2.7. (Niroomand & Rezaei, 2011)

Let $\{ x_1, ..., x_{k(G)} \}$ be a system of representatives for the conjugacy classes of a finite group G, $k(G)$, then

$$P^{\wedge}(G) = \frac{1}{|G|} \sum_{i=1}^{k(G)} \frac{\left| C_G^{\wedge}(x_i) \right|}{\left| C_G(x_i) \right|}. \tag{4}$$

Theorem 2.8. (Niroomand & Rezaei, 2011)

For every finite group G,

$$\frac{P(G)}{|M(G)|} + \frac{|Z^\wedge(G)|}{|G|}\left(1 - \frac{1}{|M(G)|}\right) \le P^\wedge(G) \le P(G) - \left(\frac{p-1}{p}\right)\left(\frac{|Z(G)| - |Z^\wedge(G)|}{|G|}\right),$$

where p is the smallest prime number dividing the order of G.

The following corollary shows that cyclic groups are characterized by the property $P^\wedge(G) = 1$ in the same way as $P(G) = 1$ characterizes abelian groups.

Corollary 2.9. (Niroomand & Rezaei, 2011)

Let p be the smallest prime number dividing the order of G. Then G is cyclic if and only if $P^\wedge(G) = 1$.

The Computation of the Exterior Degree of Some Finite Groups

In this section, we compute the exterior degree of some 2-generator p-groups of class two. The classification of these groups is given by Magidin (2006), stated in the following theorem.

Theorem 3.1. (Magidin, 2006)

Let G be a finite nonabelian 2-generator p-groups of nilpotency class two, p an odd prime. Then G is isomorphic to exactly one group of the following three types:

3.1.1. $G \cong \left\langle a,b \,\middle|\, a^{p^\alpha} = b^{p^\beta} = [a,b]^{p^\gamma} = [a,b,a] = [a,b,b] = 1 \right\rangle$ where α, β, γ are integers, and $\alpha \ge \beta \ge \gamma \ge 1$.

3.1.2. $G \cong \left\langle a,b \,\middle|\, a^{p^\alpha} = b^{p^\beta} = [a,b,a] = [a,b,b] = 1, a^{p^{\alpha-\gamma}} = [a,b] \right\rangle$ where α, β, γ are integers, and $\alpha \ge \beta$, $\alpha \ge 2\gamma$, $\beta \ge \gamma \ge 1$.

3.1.3. $G \cong \left\langle a,b \,\middle|\, a^{p^\alpha} = b^{p^\beta} = [a,b,a] = [a,b,b] = 1, a^{p^{\alpha+\sigma-\gamma}} = [a,b]^{p^\sigma} \right\rangle$ where $\alpha, \beta,$ γ, σ are integers, and $\gamma > \sigma \ge 1$, $\alpha + \sigma \ge 2\gamma$, $\alpha \ge \beta$, $\beta \ge \gamma$.

In this paper, the exterior degree is computed for the case $p = 3$ for groups of Type 3.1.1 of order less than 1000. There are seven groups that fall under this category. To compute the bound of the exterior degree, we use Groups, Algorithm and Programming Software (GAP) to first identify the groups that fulfil the classification and then find the number of conjugacy classes ($k(G)$), order of the group ($|G|$), $P(G)$, order of the center ($|Z(G)|$), order

of the exterior center ($|Z^{\wedge}(G)|$), and order of the Schur multiplier ($|M(G)|$). Next, Theorem 2.8 has been used in the computation of the bound of the exterior degree. **Table 1** shows the bound of the exterior degree for these groups.

Table 1 The exterior degree for 2-generator 3-groups of nilpotency class 2 of order less than 1000

Type	Parameters			$|G|$	$k(G)$	$P(G)$	$|Z(G)|$	$|Z^{\wedge}(G)|$	$|M(G)|$	$a \leq P^{\wedge}(G) \leq b$	
	α	β	γ							a	b
	1	1	1	27	11	11/27	3	1	6	8/81	29/81
	2	1	1	81	33	11/27	9	3	6	8/81	29/81
	3	1	1	243	99	11/27	27	9	6	8/81	29/81
3.1.1	2	2	1	243	99	11/27	27	1	9	107/2187	245/729
	4	1	1	729	297	11/27	81	27	6	8/81	29/81
	3	2	1	729	297	11/27	81	3	9	107/2187	245/729
	2	2	2	729	105	35/243	9	1	18	61/6561	89/729

From this table, it can be seen that for all nonabelian 2-generator p-groups of nilpotency class two (Type 1) of order less than 1000, $P^{\wedge}(G) \leq P(G)$.

Conclusion

The exterior degree of a group G is the probability for two elements x and y in G such that $x \wedge y = 1_{\wedge}$ where $1_{\wedge}$ is the identity of $G \wedge G$. In this paper we compute the bound of the exterior degree, $P^{\wedge}(G)$ for 2-generator 3-groups of nilpotency class two of Type 3.1.1 of order less than 1000.

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