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ASSESSING PRE-UNIVERSITY STUDENTS' VISUAL REASONING: A GRAPHICAL APPROACH

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Abstract

The importance of understanding graph in the learning of calculus had led to calls for an increased in visual reasoning skills among secondary through university levels. The effectiveness of graphs in understanding derivatives depends on their efficacy as visual text forms aimed to illustrate and communicate two or more related information. Providing students with an approach to reading and interpreting graphs will help them to understand concepts better. Therefore, the study focussed on how to assess 194 pre-university students' ability to read and interpret graphical form of functions and their derivatives based on the decoding theory: reading the graph, reading between the graph and reading beyond the graph. Findings indicate that students were able to read information directly but faced difficulties when reading beyond the graph. Implications and future research directions are discussed.

Keywords *Assessing, visual reasoning, derivative, graphical approach.*

INTRODUCTION

The new reforms in the calculus teaching and learning in the west recognized the importance of graph in the learning of calculus (NCTM, 2000). These led to calls for an increased in researches on visual reasoning skills at all levels of educations. Some educators and mathematicians had focussed on the study into the development of reasoning ability in solving mathematical problems from various approaches such as reading and interpreting graphics. Malaysia, in answering to the calls, is moving towards the implementation of visual techniques and reasoning skills among students at upper secondary schools through university levels. These approaches are able to provide encouraging effects and new insights on the teaching and learning of calculus, specifically derivatives, in the classroom (Alacaci, Lewis, O'Brien, Jiang, 2011; Larkin & Simon, 1987, Lohse, 1993). However, the issue of how to assess visual reasoning ability through these approaches are still a question to many educators. Current assessment on solving mathematical problems, such as calculus or other mathematical areas, are still based on algebraic and symbolic manipulations, procedural skills (Noraini & Lim, 2007) and even memorization in arriving to the solutions rather than venturing assessment into the students various reasoning skills such as visual reasoning (Lowrie & Diezmann, 2011; Friel, Curcio & Bright, 2001; Zimmermann, 1987). Students are very well-versed with finding the derivatives of functions algebraically be it using the chain rule, product rule, quotient rule or even implicit differentiation. Unfortunately, when asked to 'search' for the derivatives from the graph of the functions, most are still not able to establish the connections. Analyzing changes between two quantities in various contexts is a crucial educational goal for calculus curriculum (NCTM, 2000). Most textbooks and examination materials on functions and derivatives focus on reading and interpreting drawn graphs. Cartesian graphs provide visual tools to display how one variable changes with respect to another related variable and provide rich connections between functions and their derivatives (Monk, 1994; Stahley, 2011; Ubuz, 2007). In order to comprehend derivatives, students should have the abilities to read and interpret information embedded in Cartesian graphs.

Consequently, more reading and interpreting information embedded in graphs types of problems that demonstrate the strength and efficiency of visual reasoning ability should be constructed for classroom and examination practices. Therefore, it is necessary to take a closer look on how to assess students' visual reasoning through the use of graphs that focus on the integration of abilities such as reading and interpreting graphs of functions (Ratwani, Trafton & Boehm-Davis, 2008; Tiwari, 2007). The study utilized Making Sense of Graphs (MSG) theory, developed by Friel, Curcio and Bright (2001) and the Performance Standards outlined by the South Australian Certificate of Education (SACE) (2014), to assess students' visual reasoning ability. They were used to construct items with respect to the three levels of decoding process: reading the graph, reading between the graph and reading beyond the graph. The levels are presented in increasing complexity manner. Visual reasoning has been shown to be a vital ability to perceive the behaviour of functions and interpreting related derivative properties. Based on the syllabus set for the Malaysian Integrated Curriculum for Secondary Schools, differential calculus is a prerequisite at the pre-university levels while in the South Australian Certificate of Education (SACE, 2014) curriculum statements, differential calculus serves the one of the main areas of focus. In this study, various nature of students' visual reasoning ability were captured through their performance in using graphs to solve problems across five content domains: the rate of change, the tangent to the curve, the properties of functions, graphs of functions and their derivatives and the applications of derivatives. Further, this study sought to identify in-depth the students visual reasoning process in solving derivative problems through the analysis of their reasons for carrying out particular method(s). Specifically, this study addressed the following research questions:

1) What are the pre-university students' levels of visual reasoning in the use of graphs to solve derivative problems?
2) How do the pre-university students employ graphs as visual information to reason in solving derivative problems?

THEORETICAL FRAMEWORK

The MSG was designed mainly to assess students' graphs comprehending ability in the school learning context (Friel, Curcio & Bright, 2001). It had been used to analyze the critical factors influencing students processing and responses when using statistical graphs in solving mathematical problems. MSG provide framework to assess and classify the quality of reading and interpreting from graphs which can be inferred from the structure of work solutions and the reasoning provided by the students. According to the MSG, classifying a student's worked solutions can be done at three levels: reading the data, reading between the data and reading beyond the data. During the elementary phase reading the data, students investigate graphs by focussing on the extracting of data from graphs. Students are to find and locate, and translate information based on the specific rules or conditions (Murray et al., 1997). Translation requires a change in the form of a communication. To translate between words and graphs requires students to describe the specific structures of the graphs (Jollifffe, 1991; Wood, 1968). The intermediate phase of reading between the data focuses on the process of interpolating and finding connection in the data shown in the graphs. Students are to integrate or pull together two or more pieces of information (Murray et al., 1997), make comparisons and to observe relationships among the 'specifiers' or between the 'specifiers' and the labelled axes. Friel, Curcio and Bright (2001) define 'specifiers' as any form of visual dimension to represent the data values. In order to interpret the data embedded in the graphs, students need to rearrange information in order of their importance (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997). The advance phase of reading beyond the data involves the process of applying graphs that focuses on extrapolating information and analyzing the relationships implicitly out of the data shown in the graphs. Students are to generate, predict and make inferences of the data. To extrapolate, students need to extend the interpreting phase by stating not only the essence of the communication but to identify some of the consequences through noting the trend perceived in the data or specifying implications and also based on personal background knowledge (Murray et al., 1997).

Derivative in calculus is much more than a collection of concepts and skills; it is a way of approaching new challenges by investigating, modelling, reasoning, visualising, and problem-solving, with the goal of communicating to others the relationships observed and problems solved. Based on the SACE Curriculum Statement (2014), the Mathematical Studies Performance Standard outlines three main areas of measure as guide on how the students are progressing in their learning: 1) Mathematical Knowledge and Skills and Their Application (MKSA), 2) Mathematical Modelling and Problem Solving (MMP) and 3) Communication of Mathematical Information (CMI). For the MKSA, students are expected to demonstrate their knowledge of content and understanding of mathematical concepts and relationships. They are expected to use mathematical algorithms and techniques to find solutions to routine and complex problems, application of knowledge and skills to solve problems in different contexts, and the use of technology. MMP requires the development of mathematical models that lead to mathematical results, development of mathematical results for problems set in familiar and unfamiliar contexts, interpretation of mathematical results in the context of the problem, understanding of the reasonabless and possible limitations of the interpreted results, and recognition of assumptions made and possible new mathematical questions to be investigated. CMI focuses on communications of mathematical ideas and reasoning to develop logical arguments, use of appropriate mathematical notation, representation and terminology.

In this study, the structure of worked solutions and the reasoning provided in solving derivative problems graphically can be classified into three levels of decoding: reading the graph, reading between the graph and reading beyond the graph. The levels are ordered in terms of the complexity of extracting information and properties of functions and their derivatives. The theoretical framework was based on the expectation that students are able to exhibit the three levels of decoding process across the five content domains and performance standards. Table 1 shows the framework of the study on the characteristics of the students' visual reasoning ability incorporating the five content domains of functions and their derivatives.

Table 1 Visual Reasoning Ability Framework

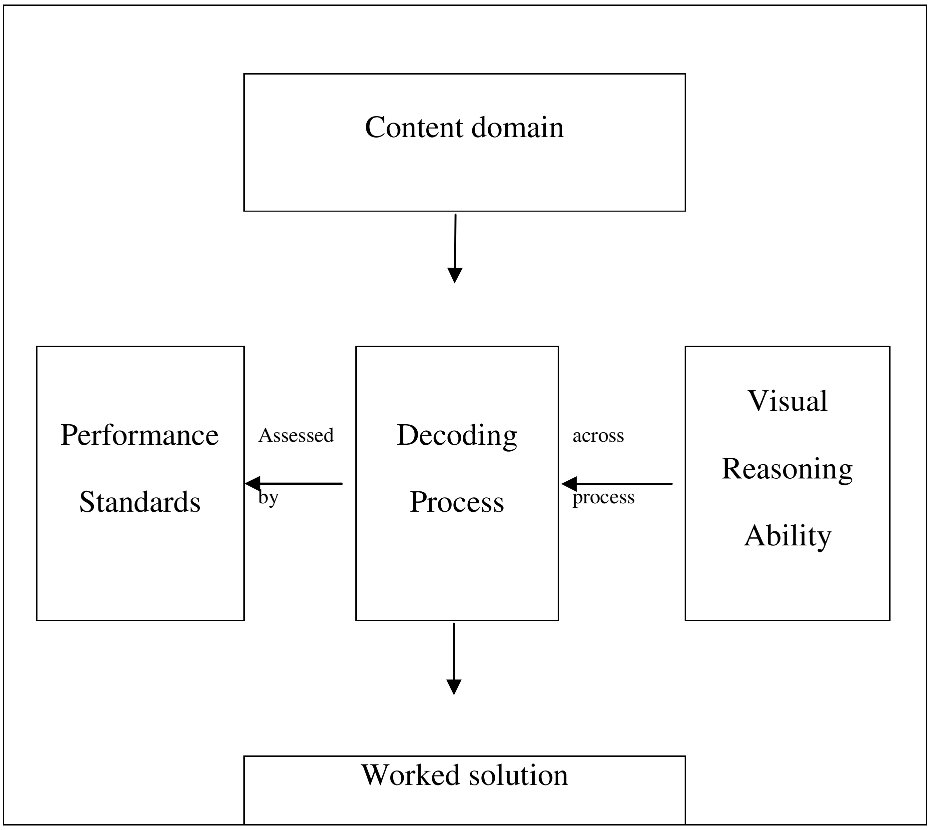
	Investigating graph (reading the data)	Interpreting graph (reading between the data)	Applying graph (reading beyond the data)
Slope / Rate of change	• Identify the y-coordinate of a given point • Identify as the increment in one variable with respect to another related variable	• Making comparison of the slopes	• Make relationships between instantaneous and average rate of change
Tangent	• Identify the coordinate of a point on the graph and line as the point of contact • The location / position of graphs of functions (one above the other)	• Calculating the slope of the tangent at the point of contact • Relationship of the two functions in terms of the distance between them	• Notice the relationship of the slope of the tangents at particular points or particular conditions
Properties of graphs	• Identify the coordinates of zeros • Read off the vertical asymptote and the horizontal asymptote	• Identify the equation of the vertical asymptote • Making connection of the zeros of y=f '(x) as the stationary points of y=f(x)	• Make decision on the nature of stationary points through the signs of y=f '(x) • Make connection of the visible shape of the graph with the signs of y=f '(x) and y=f ''(x)
Graphs of functions and their	• Identify the increasing and decreasing parts of the graph	• Identify the shapes of the graph • Evaluating y=f(x) as x	• Maki connections of the graphs and their derivatives

derivatives		positive negative infinity	
Application of derivatives	• Comparing the real life situations verbally and the graphs of the situation	• Describing the slope of the graphs as rate of change	• Identify the properties of the second derivative from the graph of y=f(x) • Relating important points and properties of the graphs to the real life situation

Figure 1 represents the theoretical framework of this study. Friel et al., (2001) and Lowrie (2011) viewed that the ability to use graphs to extract information and solve the problems involves a number of cognitive process which consists of:

1) Reading the graph by reading-off information directly as shown in the graph.
2) Reading between the graph by evaluating and making relationship among the information shown in the graph.
3) Reading beyond the graph by making deduction and conclusion on the information shown in the graph.

This study proposed that the visual reasoning ability can be assessed based on the above three processes. Five functions and derivatives problems in the form of graphical tasks were assigned to the students, allowing them to exhibit such abilities. Each items consisted of several parts in order to in-depth investigate the criteria required from the students. The correct solutions together with valid reason(s) provided by the students serve as indicator on the ability to response to the problems at particular decoding levels.



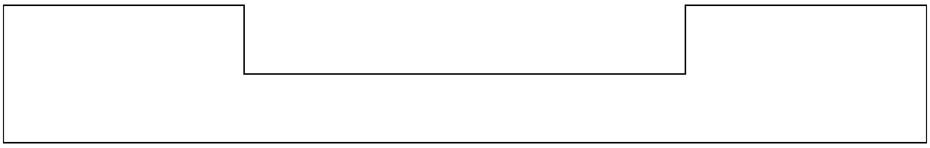


Figure 1 Theoretical framework of the study

METHODOLOGY

Participants

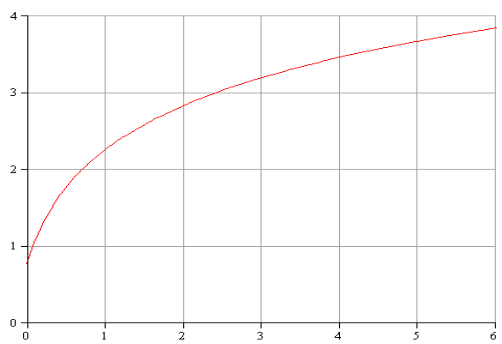
In Malaysia, the basic idea of calculus is introduced during the upper secondary school to serve as basis for the massive applications of the derivative (and differential calculus) at higher schools or pre-university levels. Graphs are then employed as visual information by either the traditional pencil-and-paper method or technology to help students understanding the concepts. Thus, assessing the students' visual reasoning was important as educators could gain greater awareness on alternative methods of teaching to enhance learning outcomes. The participants of this study were 194 pre-university students studying the South Australian Matriculation (SAM) programme in Malaysia with the intention to pursue into various disciplines at tertiary level. At the time of the study, the students had finished the SACE calculus syllabus and therefore had been exposed to the concepts and applications of derivative for at least a year period.

Instrumentation

A set of five items instrument, the Visual Reasoning Test, was employed to investigate the extent students employ graphs as visual information to reason in solving derivative problems. All tasks were graph-accompanied questions where students need to refer to the graphs for information and hence solutions. All tasks were set in accordance to the curriculum outlined by the SACE. All the items were open-ended questions. Students were requested to provide reasons or explain on the method(s) they carried out so as to solve the problems. The decoding levels were incorporated in all tasks so as to in-depth determine their visual reasoning skills. The following are examples of items designed for the study.

Reading the graph

The graph of a function $y = f(x)$ is as shown below. ***Give reason your answers***



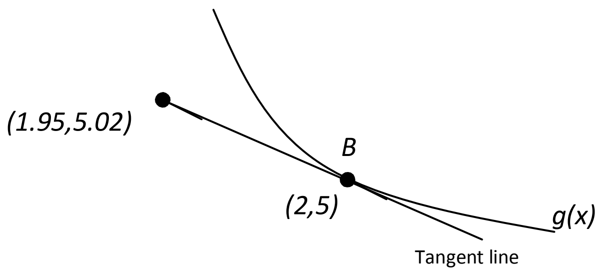
For each of the following, decide which is larger: $f(2)$ or $f(4)$

Answer: $f(4)$ is larger

Note: The item requires the students to understand the concept of function and that $f(x)$ graphically represents the location of the y-coordinate for its respective x-value. Students are able to see directly from the shape of the graph that the function is an increasing function and therefore $f(4)$ is greater than $f(2)$. Students may also argue that visually the position of $f(4)$ is higher than the position of $f(2)$.

Reading between the graph

Use the figure given below to fill in the blanks in the following statements about the function $y = g(x)$ at point B. ***Give reason to your answers.***



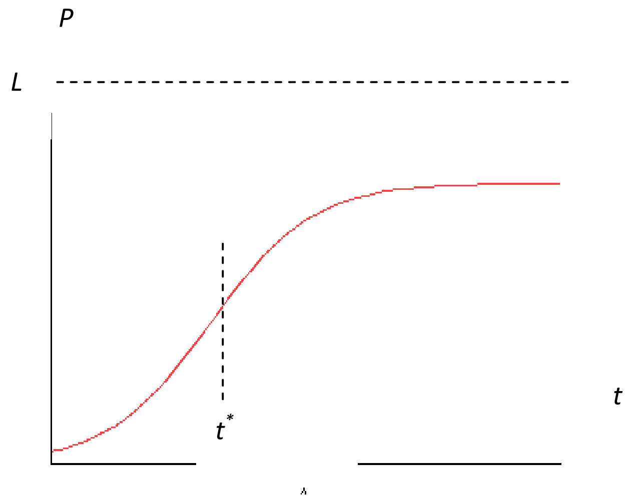
$g'(\_\_) = \_\_\_\_\_$

Answer: $g'(2) = \frac{5-5.02}{2-1.95} = \frac{-0.02}{0.05} = -0.4$

Note: The item requires students to understand the concept of derivative and tangent to the curve. They must know that the derivative of the function at point B is equal to the slope of the tangent line touching the curve at point B. Students should also notice that the tangent line is slanting downwards or a decreasing linear function and therefore the slope is negative.

Reading beyond the graph

A population, P, growing in a confined environment often follows a logistic growth curve , as shown in the diagram below. ***Give reason for your answers***.



What is the practical interpretation of t^* and L.

Answer :t^* represents the time when the population is growing at the greatest rate and L represents the limiting capacity of the population.
Note : Students can make inferences through the understanding of the relationship between the graph of the function $y = f(x)$ and its first or second derivatives, and infer them to the real life situation represented by the graph of function. They should be able to see that the slope is the steepest at point t^* and therefore represents the maximum rate of change of the population. Students are able to see that the graph does not go beyond the horizontal line y = L which infer it as the limiting value or limiting capacity of the population.

Procedure

This study was divided into two phases; the quantitative and qualitative approach. The rationale for selecting the quantitative method was the abundant of data used to assess the students' visual reasoning ability in solving derivative problems. The second phase of the study employed qualitative method to clarify the students' visual reasoning ability through the description of steps taken or reasons for carrying out particular solution process. The students completed the tasks individually in approximately 45 minutes.

Data Analysis

The data analysis had been carried out based on the students' solutions and reasoning of the tasks. In this study, we use the term visual reasoning to refer to the handling of visual elements in making and searching for spatial relations to deduce the related conceptual understanding. We examined the use of graphs to evaluate how functions and their derivatives are related and depending on each other. By investigating worked solutions through the use of graphs, we sought to understand how to assess students' visual reasoning skills. Only three of the items are discussed in this article.

The first phase of the analysis was subjected to frequency and percentage of the responses based on five ordered performance category levels : 0 = the students did not attempt or left the task unanswered, 1 = the student produced incorrect solution together with incorrect reason or did not provide any reason at all, 2 = the student managed to arrived to the correct solution but did not provide any reason for it, 3 = the student managed to arrive to the correct solution but had provided invalid reason, 4 = the student managed to arrive to the correct solution together with valid reason, as shown in Table 1. The second phase of the analysis involves students' qualitative information about their reasons or methods they employed to arrive to the solutions so as to further investigate their thinking, understandings or misconceptions about the graphs.

Table 2 General rubric for the Visual Reasoning Test

Point	Code	Description
4	CSVR	Correct solution with valid reason
3	CSIR	Correct solution with invalid reason
2	CSNR	Correct solution with no reason
1	ISINR	Incorrect solution with invalid reason / no reason
0	NA	No answer / Not attempted

RESULTS

Quantitative results

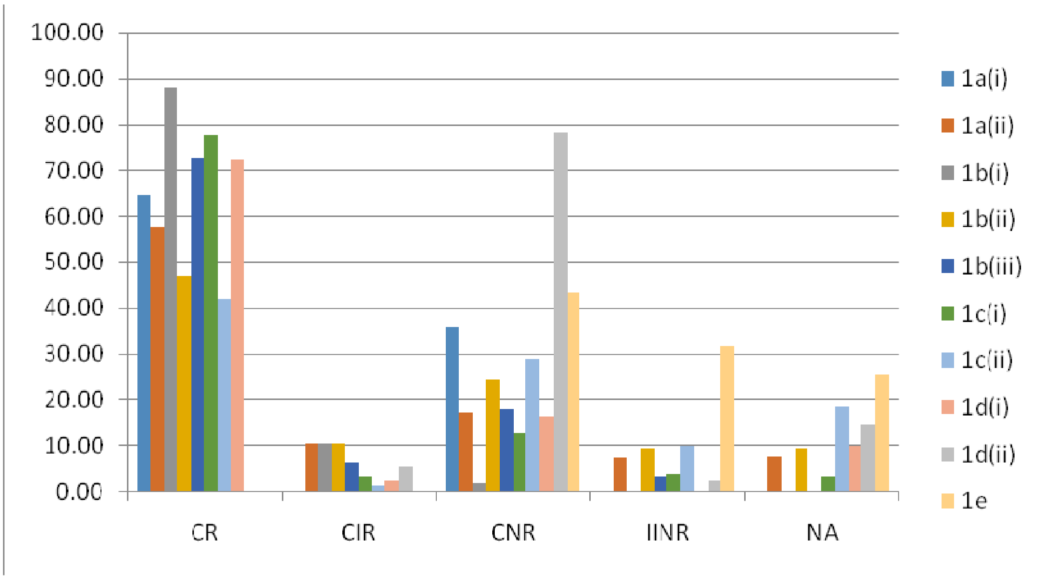


Figure 2 The percentage distribution of responses item 1

The first research question looks for the students' levels of visual reasoning ability. Figure 2 displays the percentages of students' types of solutions for item 1. The analysis shows at least 40% of the students managed to get correct solutions together with correct reasoning while less than 10% of the students produced correct solutions but had reasoned them wrongly. A mix portion of students produced correct solution but failed to supply any reason for them. The results indicate that students were able to read the information embedded in the graph and reason them. Smaller portions of less than 20% of the students either came out with incorrect reasons or did not attempt the tasks at all.

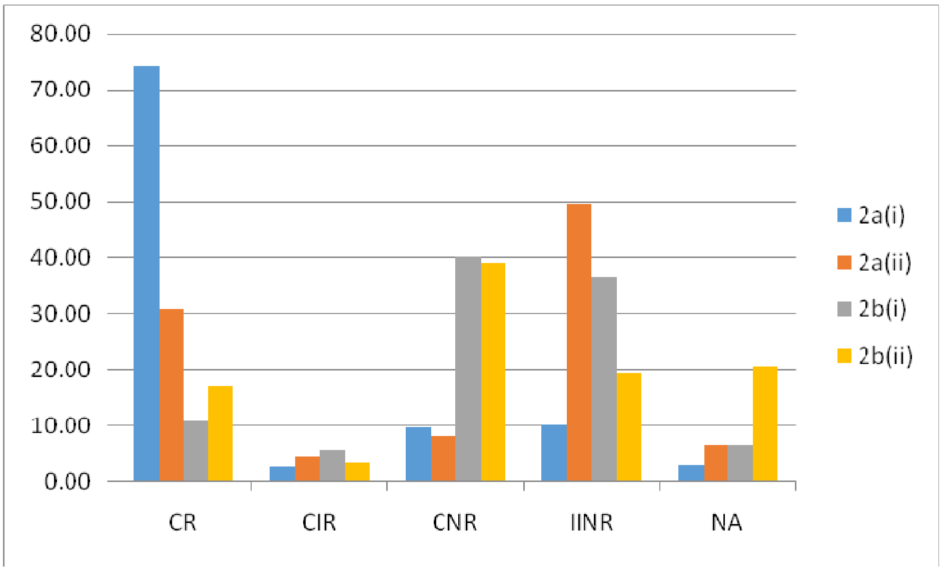


Figure 3 The percentage distribution of responses for Item 2

Figure 3 displays the percentages of the types of students' solution for item 2. A total of approximately 86% of the students managed to solve the problem 2(a)(i) correctly regardless whether they provided correct reason or incorrect reason while a total of less than half (43.82%) of the students were in the same category for task 2(a)(ii). More than half of the students either produced incorrect solutions with incorrect or no reason, or did not attempt the task at all. The results for items 2(b)(i) and 2(b)(ii) show that bigger percentages of the students were unable to provide reasons although they managed to get the correct solutions for the tasks while others produced incorrect solutions. The analysis indicate that as the tasks are getting more complicated and require deeper analytical thinking, the lower the percentages of students who are able to solve them.

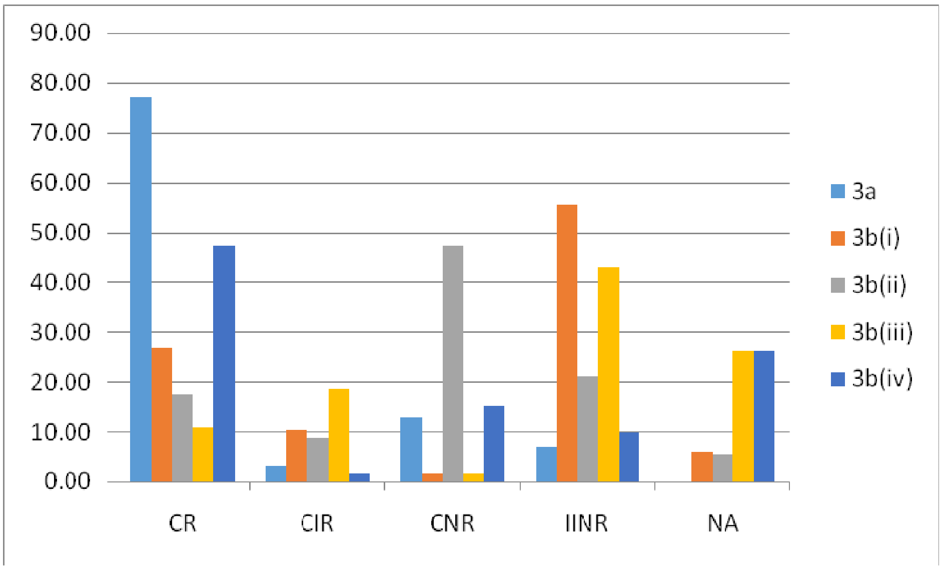


Figure 4 The percentage distribution of responses for Item 2

Figure 4 displays the percentages of the types of students' solution for item 3. More than 50% of the students managed to obtained correct solutions for items 3(a), 3(b)(ii) and 3(b)(iv) although some percentages of them came out with either incorrect reasons or did not provide any reason for their solutions. Again as the tasks getting 'harder' (as items 3(b)(i) and 3(b)(iii)) the students were prone to produced incorrect solutions and providing incorrect reasons or did not even attempt the questions. The analysis shows mix types of solutions and consequently reflects the fluctuating levels of visual reasoning ability of the students.

Qualitative results

At the end of each task, students were requested to provide reasons or methods they employed in order to reach to the solutions. Samples of extracts for valid and invalid reasons provided by the students to items in particular categories are as shown in Appendix 2. Those students who managed to solved and reason correctly were able to read off and interpret data from the graphs by referring to the information displayed or embedded in the graphs. They tend to have strong foundation of and understand the concepts of functions and their derivatives, and managed to show their high ability in visual reasoning skills.

The problems of lack of understanding and conceptual knowledge of functions and their derivatives had led students to produce incorrect reasons, regardless whether they had managed to arrive to the correct solution or incorrect solutions. Reasons such as 'tangent is larger', 'tangent is positive' and 'tangent between two points' exhibit students' lack of communication skills and weak in the use of appropriate mathematical terminologies. Some responses were as brief as 'increasing and decreasing' only and other various incorrect descriptions of 'rate of change of population' indicate that students failed to relate the slope of the function to the rate of change of the function representing real life situation as population.

IMPLICATIONS AND CONCLUSION

Students, at all levels have always portrayed calculus, specifically differential calculus as one of the most difficult topics in mathematics. They often misunderstood the notion of function, it properties and the related derivative graphically. Most of the students are able to easily manipulate the algebraic rules of differentiation, be it the chain rule, product rule, quotient rule, or even implicit differentiation. The visual approach seems to be manageable when they need to just read off information but as the tasks get complicated, majority of the students tend to perform incorrect reading-off or interpretation of the graph. Researchers had suggested that the more complicated the tasks assigned to the students, the more cognitive effort are required in order to understand the information embedded in the graphs or diagrams (Uesaka and Manalo, 2011; Sharma, 2013; Alacaci, Lewis, O'Brien & Jiang, 2011).

The results also indicated that the students were having problems in describing their reasoning or methods they carried out to arrive to the solutions. One contributing factor is due to the rare use and understanding of standardized mathematical terminologies. Some are still reverting to the mother-tongue language (Malay, in this study) when trying to understand more complicated concepts. Another contributing factor is the teaching that focused on procedural rather than conceptual knowledge. Students tried to memorize formulae and facts that lead them to have vague understanding of the concepts and consequently, were not able to express their reasoning (Noraini, 2008; Parmjit, 2006). Teachers are to modify their method of teaching by using more visually-represented materials in the classroom so as to gain students' attention and to make understanding of concepts easier as agreed by Sharma (2013) and Watson (2006). The curriculum developers can also make use of the results by including more graph-based or diagram-based approaches starting from the lower level and throughout all levels of educations.

To summarize, our study revealed a mixture of students' visual reasoning ability. It is clear from the results of the study that most of the students were very competent in reading information directly from the graphs, but faced some difficulties as the tasks get more complicated. We conjecture that appropriate understanding of reading and interpreting graphs is dependent on their conceptual knowledge of functions and derivatives. In our opinion, this situation is not a sign of disability to reason visually but also a sign of weaknesses in the content domains. Therefore, they are likely in need to be exposed to alternative methods of understanding concepts and classroom instructional. Much attention should be focussed to cultivate student's ability on the use of graphs as visual reasoning tools. If this is not implemented, students are likely to progress in complacence with their procedural method of solution (especially those who manage to arrive to the correct solution) and may not tolerate in their mathematical problem solving experiences at the higher level of educations. It is hoped that the findings of the study will generate more interest in the research area and to expand on the literature of students' visual reasoning ability and conceptual understanding of functions and their derivatives and to enhance our awareness of students' thought processes and visual reasoning skills. There is also a need to look into the students' ability to construct appropriate and detailed graphs and diagrams to solve particular problems. Pre-university students' lack of skills to reason visually raises important issues to educators and mathematics education.

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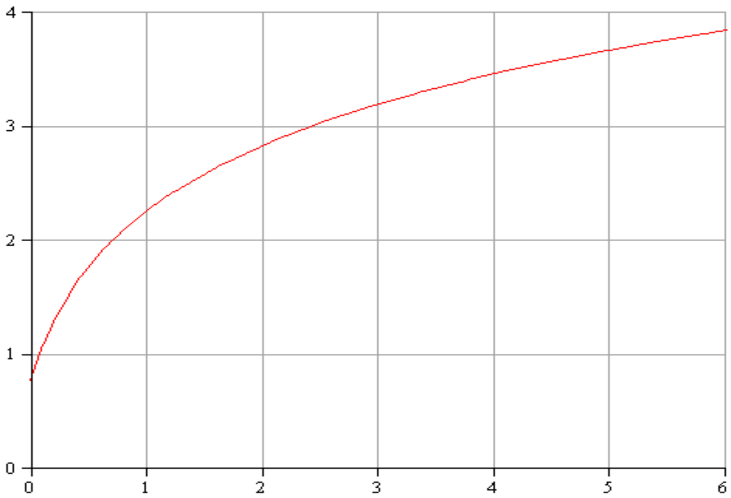
APPENDIX 1

Task 1

The graph of a function $y = f(x)$ is as shown below. ***Give reason your answers***.



(a) Represent the following on the graph.

(i) $f(4)$ (*with a letter 'P'*)

(ii) $\dfrac{f(3)-f(1)}{3-1}$ (*with a letter 'Q'*)

(b) For each of the following, decide which is larger.

(i) $f(2)$ or $f(4)$

(ii) $\frac{f(2)-f(1)}{2-1}$ or $\frac{f(4)-f(3)}{4-3}$

(iii) $f'(1)$ or $f'(4)$

(c) Circle the correct answer :

(i) $f'(1)$ = positive or negative

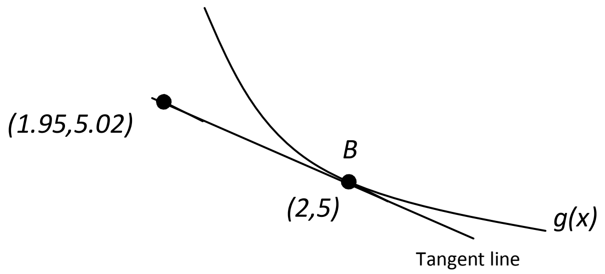
(ii) $\frac{f(x)-f(1)}{x-1}$, $2 \le x \le 3$ = positive or negative

(d) Illustrate both (c)(i) and (c)(ii) graphically (on the graph above).

(e) Write down the relationship between (a)(i) and (a)(ii).

Task 2

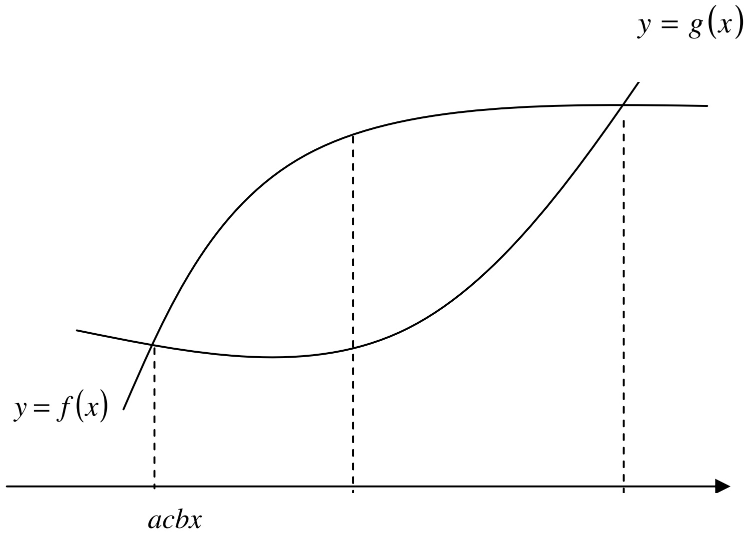
(a) Use the figure given below to fill in the blanks in the following statements about the function $y = g(x)$ at point B. ***Give reason your answers.***



(i) g (__) = ______

(ii) g′ (__) = ______

(b)



Let $f(x)$ and $g(x)$ be the differentiable functions graphed above.

(i) Find the expression for the vertical distance, $d(x)$, between the two curves.

Point c is the point where the vertical distance between the curves is the greatest.

(ii) Is there anything special about the tangents to the curves at c?

Give reason(s) for your answer. Explain all steps taken.

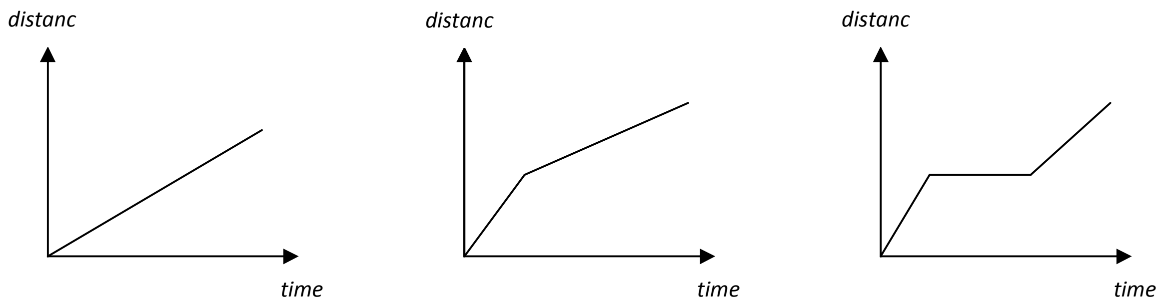
Task 3

(a) There are three routes from Town X to Town Y.
Match the route descriptions to the appropriate distance-time graphs :

Route A : Two-lane highway direct with maximum speed limit of 110 km/hour. Thirty-minute wait at bridge-works.

Route B : Winding mountain road with steep gradients and curves requiring you to travel at a constant slower speed.

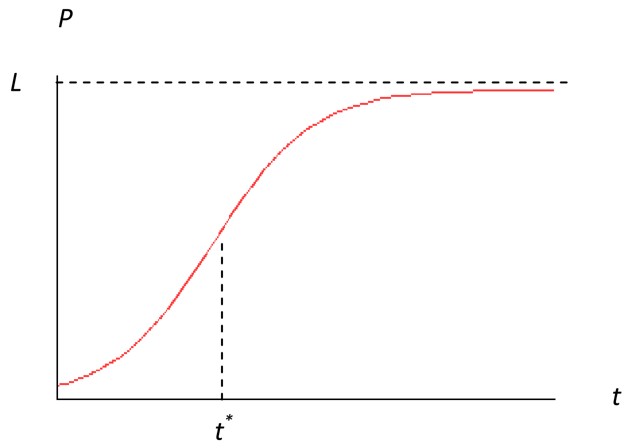
Route C : Two-lane highway with maximum speed limit of 110 km/hour and then winding detour to avoid bridge-works.



Route _____ Route _____ Route_____

Explain your reasons in making the decisions.

(b) A population, P, growing in a confined environment often follows a logistic growth curve , as shown in the diagram below. ***Give reason for your answers***.



(i) Describe how the *rate* at which the population is increasing cha ver time.

(ii) Draw the sign diagram for the second derivative, $\frac{d^2P}{dt^2}$.

(iii) What is the practical interpretation of t^* .

(iv) What is the practical interpretation of L.

APPENDIX 2

Reading the graph

Item	Correct response	Incorrect response
1(b)(i)	Increasing function	Logarithmic function
2(a)(i)	The x and y values of function	Minimum point

Reading between the graph

Item	Correct response	Incorrect response
1(a)(ii)	The slope of a line joining two points	A tangent
1(b)(ii)	One line is steeper than the other line	Tangent is larger
1(b)(iii)	One tangent line is steeper than the other tangent line	Higher point
1(c)(i)	The line is slanting to the right	Tangent is positive Negative slope
1(d)	A tangent and a chord	A tangent Slope between two points
2(a)(ii)	Slope of tangent at a point	Tangent to the curve Stationary point Negative slope

2(b)(i)	The higher function minus the lower function	Difference between the two functions Functions are equal at x=a and x=b Function intersect at two points
3(a)	The slope represent the speed	Shape of graph Passes through the origin
3(b)(i)	The slope of the graph represent the rate	Various incorrect description of the function : • Increasing continuously without bound • Increase then decrease
3(b)(ii)	The shape of the curve represent the second derivative	Increasing and decreasing Function increasing but starts to slow down No / wrong critical point Sign diagram of dP/dt Does not cut the x-axis

Reading beyond the graph

Item	Correct response	Incorrect response
1(c)(ii)	The line is slanting to the right	Slope of chord Tangent between two points Negative slope
1(e)	Chord becomes tangent as distance approaching zero	Inverse function One is the slope of the other
2(b)(ii)	Slopes are equal	Difference between the two functions Functions are equal at x=a and x=b Function intersect at two points
3(b)(iii)	Maximum slope	Change shape Inflexion point
3(b)(iv)	Horizontal asymptote	Approaching positive infinity Graph is below the horizontal asymptote